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MATHEMATICS.—*On the empirical representation of certain production curves.*[1] C. E. VAN ORSTRAND, Geological Survey.

INTRODUCTION

Production data representing the yearly output from individual mines, or from groups of mines included within areas of the order of magnitude of states or nations, can generally be represented by a curve (*a*, figure 1) which has a zero value at the origin of time—the curve rises irregularly and oftentimes quite abruptly to a maximum value and then declines rather slowly; presumably to a zero value, in an interval of time which may be assumed to be finite, or infinite. The problem to be solved consists of two distinct parts—(a) the selection of an empirical equation, and (b) the evaluation of the constants of the empirical equation selected to represent the observations. It is proposed to treat both of these problems with sufficient detail and generality to enable the investigator to adopt at the outset certain well defined methods of procedure that will enable him to obtain satisfactory results with a minimum expenditure of time and energy.

SELECTION OF AN EMPIRICAL EQUATION

Expressed in terms of mathematics, we may say that the equation,

$$y = \mathrm{F}(x)$$

which represents the production (y) in terms of the time (x) has a zero value at the origin and at infinity—between these two values of the abscissae, the function is finite, positive, and continuous. The first derivative vanishes at infinity, but it may, or may not, vanish at the origin.

[1] Published with the permission of the Director of the U. S. Geological Survey.

Functions satisfying these conditions can be defined in accordance with the theorems of the modern theory of functions. The following classification is based on the treatise by Forsyth.[2]

CHARACTER OF SINGULARITY		$y = F(x)$	LIMITING CONDITIONS
Accidental	(1)	$\frac{x^m(a_0 + a_1x + \cdots a_nx^n)}{(b_0 + b_1x + \cdots b_px^p)}$	$m + n < p$
Essential	(2)	$af(x)e^{-g(x)}{}_{e}{}^{-g_1\left(\frac{1}{x}\right)}$	$g(x) > 0$
	(3)	$\left[af(x)e^{g(x)}{}_{e}{}^{g_1\left(\frac{1}{x}\right)} + P\left(x, \frac{1}{x}\right)\right]^{-1}$	$g_1\left(\frac{1}{x}\right) > 0$
Accidental and essential	(4)	$ax^m e^{-g(x)}$	$m > 0\ g(x) > 0$
	(5)	$\left[ax^{-m}e^{g(x)} + P\left(x, \frac{1}{x}\right)\right]^{-1}$	$m > 0\ g(x) > 0$
	(6)	$ax^{-m}e^{-g_1\left(\frac{1}{x}\right)}$	$m > 0\ g_1\left(\frac{1}{x}\right) > 0$
	(7)	$\left[ax^m e^{g_1\left(\frac{1}{x}\right)} + P\left(x, \frac{1}{x}\right)\right]^{-1}$	$m > 0\ g_1\left(\frac{1}{x}\right) > 0$
	(8)	$\cdots \frac{P + G}{P_1 + G_1} \cdots\cdots$	Order of numerator < order of denominator

Equation (8) consists of polynomial and transcendental expressions. The functions, $f(x)$ and $P(x, \frac{1}{x})$, are polynomials, but $g(x)$ and $g_1(\frac{1}{x})$ may be either polynomial or transcendental. The exponents in $f(x)$ are unrestricted—they may be positive or negative, integral or fractional. The order of contact at the vanishing points is finite or infinite according as the singularity is accidental or essential. The inequalities tabulated in the column "Limiting Conditions" are satisfied when the first derivative vanishes at the origin. Equations V, VIII, and IX of the following list are very special cases,

$$y = a\left(1 + \frac{x}{c_1}\right)^{m_1}\left(1 - \frac{x}{c_2}\right)^{m_2} \cdots \frac{m_1}{c_1} = \frac{m_2}{c_2} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (I)$$

[2] A. R. FORSYTH. *Theory of functions of a complex variable.* Cambridge University Press, 1900. A. L. DAY and C. E. VAN ORSTRAND. *The black body and the measurement of extreme temperatures.* Astrophysical Jour. 19: 24–30. 1904.

$$y = a\left(1 - \frac{x^2}{c_1^2}\right)^m \quad \text{(II)}$$

$$y = a\left(1 + \frac{x^2}{c_1}\right)^{bc_1} e^{-bx} \quad \text{(III)}$$

$$y = a\left(1 + \frac{x^2}{c_1^2}\right)^{-m} e^{-b \tan^{-1}\frac{x}{c_1}} \quad \text{(IV)}$$

$$y = ax^{-m} e^{-\frac{b}{x}} \quad \text{(V)}$$

$$y = a(x - c_1)^{m_1} x^{-m_2} \quad \text{(VI)}$$

$$y = ae^{-bx^2} \quad \text{(VII)}$$

$$y = a[x^{-m}(e^{\frac{b}{x}} - 1)]^{-1} \quad \text{(VIII)}$$

$$y = ax^m e^{-bx^n} \quad \text{(IX)}$$

$$y = ax^m e^{-\frac{b}{\left(1 - \frac{x}{c_1}\right)^n}} \quad \text{(X)}$$

respectively, of equations 6, 7, and 4 of the preceding list. With the exception of equation VII, the remaining equations in the list are easily obtained in a similar manner by introducing zeros at the points c_1 and c_2 instead of the points 0 and ∞. Both zeros of equation VII are at an infinite distance from the origin.

Apart from certain relations between the constants, equations I to VII, inclusive, represent Pearson's seven types of frequency distribution which have been used with great success by himself and others,[3] chiefly in the field of biometrics. If we properly define the constants in IX and put $m = n = 2$, the equation represents the distribution of molecular velocities in a gas. Putting $m = 0$, $n = 2$, the equation reduces to Pearson's type VII, which is the well known normal, Laplacean, or Gaussian distribution of frequencies on which the classical theory of probabilities is based. Pearson's type III and V are likewise special cases of IX. Two other very special cases of IX,

$$y = ax^{-m} \text{ and } y = ae^{-bx}$$

are used to represent decline or extinction curves. These relations

[3] W. PALIN ELDERTON. *Frequency-curves and correlation.* C. and E. Layton, London.

are easily represented by straight lines on logarithmic paper.[4] Equation VIII contains as a special case, the equation deduced by Planck to represent at a particular temperature the distribution of energy in the spectrum of a black body.

Pearson has developed criteria based on the method of moments by means of which one may select from his list of seven functions the particular equation of the group which will most accurately represent the observed values. Unfortunately, this method of procedure, at least in its present state of development, is not easily adapted either to the determination of criteria or to the evaluation of the constants in the selected equation when the observed frequency curve is incomplete. In view of these defects and the fact that theoretical considerations give us practically no information as to the nature of the required function, it becomes necessary to adopt a different method of procedure. Evidently, variation, or degree of flexibility, or generality of functions can be obtained in only two ways—either by increasing the number of constants, or by putting the necessary mechanism into the function itself. An extreme case of the first is a Fourier series—of the second, a function of functions such as the expressions obtained for the representation of a frequency distribution by the method of semi-invariants. From the preceding brief discussion, it appears that equation IX best meets the requirement of possessing a very considerable range of generality expressed in terms of a minimum number of constants. It has therefore been selected for a first trial.

COMPARISON OF METHODS OF EVALUATION OF CONSTANTS OF FREQUENCY DISTRIBUTIONS

The precise evaluation of the constants of a frequency distribution oftentimes presents more difficulties than does the selection of an equation which is assumed to represent the data of observation. Four methods of evaluating the constants are now in use: (1) Method of least squares; (2) Pearson's method of moments;[3] (3) Thiele's[5] method

[4] CHARLES S. LARKEY. *Mathematical determination of production decline curves.* Mining and Metallurgy **4**: 341–344. 1923. HARRY M. ROESER. *Determining the constants of oil-production decline curves.* Trans. Am. Inst. Mining and Metallurgical Engineers. (Birmingham meeting, October, 1924.) WILLARD W. CUTLER, JR. *Estimation of underground oil reserves by oil-well production curves.* Bull. 228 U. S. Bureau of Mines, VII + 114 pages, 1924.

[5] ARNE FISHER. *An elementary treatise on frequency curves.* The Macmillan Co. 1922. ARNE FISHER. *The mathematical theory of probabilities.* Vol. 1. Second edition. 1923. The Macmillan Co.

of semi-invariants; and (4), Krichewsky's[6] method of evaluating the constants of the first derivative of the required integral equation.

Columns 3 and 4 of table 1 give Elderton's[7] computed values (y_c), and residuals (v) for the data tabulated in columns 1 and 2. The values were obtained by adjusting Pearson's type I by the method of moments. Columns 6 and 8 give the computed values obtained by adjusting equations IX and I, respectively, by the method of least squares. The computed values in column 6 are too large over that portion of the curve which approaches the age axis as an asymptote, but over the remaining portion of the curve, the residuals obtained by this method are less than those obtained by the method of moments. Using the arbitrary values, $c_1 = 9.1$, $c_2 = 65.0$, and adjusting the remaining constants of equation I by the method of least squares, we obtain the numbers tabulated in column 8. Judged by the usual criterion that the sum of the squares of the residuals (Σv^2) must be a minimum, the values tabulated in table 1, namely, 298, 550, and 169, show that the method of least squares gives a better adjustment than does the method of moments; furthermore, equation IX gives a graduation only slightly inferior to that of equation I.

Column 11 gives the values calculated by Fisher[8] from the equation,

$$y = 1130\,[\phi_0(x) + 0.0258\,\phi_0^{iii}(x) + 0.0158\,\phi_0^{iv}(x)]$$

wherein

$$\phi_0(x) = \frac{1}{1.624\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x + 0.0193}{1.6240}\right)^2}$$

and the remaining terms are respectively, the third and fourth derivatives of $\phi_0(x)$. The constants were evaluated by the method of semi-invariants from the data of columns 9 and 10. In column 13 are tabulated the values obtained from a least square adjustment of equation I, using the arbitrary values, $c_1 = c_2 = 31.0$. The values of m_1 and m_2 were assumed in this case to be independent. Comparison of the corresponding values of Σv^2, 124 and 639, suggests that it is probably impossible to adjust equation I, either by the method of least

[6] S. KRICHEWSKY. *A method of curve fitting.* Ministry of Public Works, Egypt. Physical Department Paper No. 8. Government Press, Cairo, 1922. E. W. WOOLARD. *On Krichewsky's method of fitting frequency curves.* Monthly Weather Review **52**, no. 2: 91–94. 1924.

[7] W. PALIN ELDERTON. Loc. cit. pp. 54–58.

[8] A. FISHER. Loc. cit. First reference, pp. 48–52.

squares or by the method of moments so as to obtain the remarkable degree of precision obtained in this particular instance by the method of semi-invariants.

TABLE 1.—COMPARISON OF METHOD OF LEAST SQUARES WITH METHOD OF MOMENTS AND METHOD OF SEMI-INVARIANTS

CENTRAL AGE	EXPOSED TO RISK	MOMENTS		IX		I		CENTRAL AGE	NUMBER PENSIONERS	INVARIANTS	I	
	y_o	y_c	v	x	y_c	x	y_c		y_o	v_c	x	y_c
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
17	34	44	−10	1	43	−9	43	37	1	1	−29	0
22	145	137	+8	6	137	−4	145	42	6	5	24	0
27	156	149	+7	11	158	+1	153	47	17	17	19	8
32	145	142	+3	16	149	6	143	52	48	48	14	45
37	123	127	−4	21	128	11	127	57	118	118	9	129
42	103	108	−5	26	105	+16	107	62	224	226	−4	229
47	86	88	−2	31	83	21	87	67	286	291	+1	280
52	71	69	+2	36	65	26	68	72	248	241	6	240
57	55	51	+4	41	49	31	50	77	128	126	11	139
62	37	36	+1	46	37	36	35	82	38	44	16	48
67	21	24	−3	51	28	+41	22	87	13	15	+21	7
72	13	14	−1	56	20	46	13	92	2	3	26	0
77	7	7	0	61	15	51	6	97	1	1		
82	3	3	0	66	11	56	2					
87	1	1	0	71	8	61	0					
Σy_o		1000		1000		1000			1130		1130	
Σy_c		1000		1036		1001			1136		1125	
Σv^2		298		550		169			124		639	
$\log_e a$		0.94440		3.83910		5.03218			....		5.62412	
m		....		0.86077		....			....		....	
b		....		0.07650		....			....		....	
m_1		0.40983		....		0.35530			....		7.00455	
m_2		2.77698		....		2.53785			....		6.42181	
c_1		1.99638		....		9.1			....		31.0	
c_2		13.52728		....		65.0			....		31.0	
Origin—age		26.75942		16.0		26.0			....		66.0	
Mode		0.00000		11.25		00.0			....		1.345	
Mode—age		26.75942		27.25		26.0			....		67.345	

The method of least squares frequently fails to give a satisfactory adjustment of Pearson's type V; after passing the mode, the computed curve generally falls considerably above the observed curve.

TABLE 2.—PETROLEUM PRODUCTION IN WEST VIRGINIA, 1890–1923

TIME	PRODUCTION	IX		IX			III			I	
x	y_0	y_c	v	x	y_c	v	x	y_c	v	y_c	v
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
years	*millions of barrels*										
1	2.4	3.2	−0.8	0.5	2.6	−0.2	−11.5	2.4	0.0	2.5	−0.1
2	3.8	5.6	−1.8	1.5	5.4	−1.6	10.5	5.4	−1.6	5.5	−1.7
3	8.4	7.6	+0.8	2.5	7.6	+0.8	9.5	7.6	+0.8	7.6	+0.8
4	8.6	9.1	−0.5	3.5	9.2	−0.6	8.5	9.2	−0.6	9.3	−0.7
5	8.1	10.4	−2.3	4.5	10.5	−2.4	−7.5	10.5	−2.4	10.5	−2.4
6	10.0	11.4	−1.4	5.5	11.5	−1.5	6.5	11.5	−1.5	11.5	−1.5
7	13.1	12.1	+1.0	6.5	12.2	+0.9	5.5	12.2	+0.9	12.2	+0.9
8	13.6	12.7	+0.9	7.5	12.7	+0.9	4.5	12.7	+0.9	12.7	+0.9
9	13.9	13.0	+0.9	8.5	13.1	+0.8	3.5	13.1	+0.8	13.0	+0.9
10	16.2	13.3	+2.9	9.5	13.3	+2.9	−2.5	13.3	+2.9	13.3	+2.9
11	14.2	13.4	+0.8	10.5	13.4	+0.8	−1.5	13.4	+0.8	13.4	+0.8
12	13.5	13.4	+0.1	11.5	13.4	+0.1	−0.5	13.4	+0.1	13.4	+0.1
13	12.9	13.4	−0.5	12.5	13.3	−0.4	+0.5	13.3	−0.4	13.3	−0.4
14	12.6	13.2	−0.6	13.5	13.1	−0.5	+1.5	13.1	−0.5	13.1	−0.5
15	11.6	13.0	−1.4	14.5	12.9	−1.3	+2.5	12.9	−1.3	12.9	−1.3
16	10.1	12.7	−2.6	15.5	12.7	−2.6	3.5	12.7	−2.6	12.7	−2.6
17	9.1	12.4	−3.3	16.5	12.4	−3.3	4.5	12.4	−3.3	12.4	−3.3
18	9.5	12.1	−2.6	17.5	12.0	−2.5	5.5	12.0	−2.5	12.1	−2.6
19	10.7	11.8	−1.1	18.5	11.8	−1.1	6.5	11.7	−1.0	11.7	−1.0
20	11.8	11.4	+0.4	19.5	11.3	+0.5	+7.5	11.3	+0.5	11.4	+0.4
21	9.8	11.0	−1.2	20.5	10.9	−1.1	8.5	10.9	−1.1	11.0	−1.2
22	12.1	10.6	+1.5	21.5	10.6	+1.5	9.5	10.6	+1.5	10.6	+1.5
23	11.6	10.2	+1.4	22.5	10.2	+1.4	10.5	10.2	+1.4	10.2	+1.4
24	9.7	9.8	−0.1	23.5	9.8	−0.1	11.5	9.8	−0.1	9.8	−0.1
25	9.3	9.4	−0.1	24.5	9.4	−0.1	+12.5	9.4	−0.1	9.4	−0.1
26	8.7	9.0	−0.3	25.5	9.0	−0.3	13.5	9.0	−0.3	9.0	−0.3
27	8.4	8.6	−0.2	26.5	8.6	−0.2	14.5	8.6	−0.2	8.6	−0.2
28	7.9	8.2	−0.3	27.5	8.2	−0.3	15.5	8.2	−0.3	8.2	−0.3
29	8.3	7.8	+0.5	28.5	7.9	+0.4	16.5	7.9	+0.4	7.8	+0.5
30	8.2	7.5	+0.7	29.5	7.5	+0.7	+17.5	7.5	+0.7	7.5	+0.7
31	7.8	7.1	+0.7	30.5	7.2	+0.6	18.5	7.2	+0.6	7.1	+0.7
32	7.0	6.7	+0.3	31.5	6.8	+0.2	19.5	6.8	+0.2	6.7	+0.3
33	6.4	6.4	+0.0	32.5	6.5	−0.1	20.5	6.5	−0.1	6.4	0.0

This defect can be corrected in part by substitution of equation X for equation V.

The preceding examples serve to call attention to some of the difficulties involved in obtaining the best graduation of a particular set of observations; none of the methods is infallible, and ordinarily, the best method of meeting the requirements of a particular problem can be found only by trial and investigation.

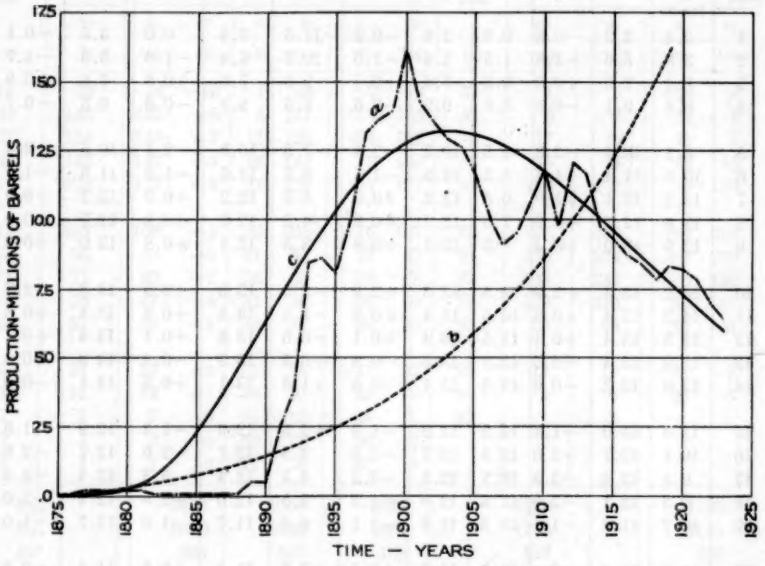

Fig. 1. Curves showing the effect of weighting the observation equations. Curve c is weighted; curve b is not weighted.

ADJUSTMENT OF SOME PETROLEUM PRODUCTION DATA

The annual production of petroleum[9] for the years 1875–1923, inclusive, for the state of West Virginia is shown graphically in figure 1. The numerical values, expressed in terms of millions of barrels, are tabulated in column 2 of table 2.

In figure 1 is shown an anomaly of the least square adjustment. Curve b was obtained by adjusting equation IX, $n = 1$, on the basis of equal weights of the observation equations written in the form

$$\log_e a + \log_e x\, m - xb = \log_e y.$$

[9] G. B. RICHARDSON. Petroleum in 1919–1921. *Mineral Resources of the United States*, 1921. Part II, pp. 253–333, May 26, 1923.

Curve c shows the result of adjusting the same equation after assigning to each observation equation the theoretically correct weight,[10] y^2. Following are the values of the constants:

Curve b. Weight 1. $\log_e a = -1.4008, m = 1.2776, b = -0.0376.$
Curve c. Weight y^2. $\log_e a = -5.6961, m = 4.5267, b = +0.1609.$

The first value of b is negative instead of positive as it should be. Attention is called to this peculiar result as it occurs quite frequently in attempting to adjust the complete data of production curves and may be incorrectly attributed to an error in the computation. The defect disappears if the origin is displaced sufficiently to the right as the following results, obtained by displacing the origin to the year 1889, readily show.

Weight 1. $\log_e a = 0.19062, m = 1.45976, b = 0.10375, \Sigma v^2 = 73.41.$
Weight y^2. $\log_e a = 0.80217, m = 1.14872, b = 0.08857, \Sigma v^2 = 65.61.$

Comparison of the values of Σv^2 shows that the best results are again obtained by properly weighting the observation equations. Equal weights may be assigned to observation equations written in the form

$$y\frac{da}{a} + y \log_e x \, dm - xy \, db = dy = v \quad \ldots\ldots\ldots\ldots \quad (1)$$

Using the preceding values of the constants obtained by assigning the weight, y^2, and solving for da, dm, and db on the basis of equality of weights, we obtain the following results:

Weight 1. $\log_e a = 0.76175, m = 1.16496, b = 0.08935, \Sigma v^2 = 65.06.$

The value of $\Sigma v^2 = 65.06$ is practically the same as the former value 65.61; consequently, the two methods lead to identical results provided the values of the constants used in the computation of $dy = v$ in the differential equation are a sufficiently close approximation to the correct values.

The true value of the exponent, n, in equation IX is not easily determined. In order to obtain distinct types of equations which are easily capable of interpretation in terms of the theory of generalized probability, Pearson adopted the integral values, $+1$, -1, and $+2$. Fractional values of n must be admitted, however, if one desires to

[10] O. M. LELAND. *Practical least squares*. First edition. p. 136, McGraw-Hill Book Co. 1921.

minimize Σv^2. Curves corresponding to various values of n are shown in figure 2. All negative values, including the value $n = -1$ for which the curve is shown in the figure, must be excluded for the reason that in all of these cases the first derivative vanishes at the origin. Comparison of the values of Σv^2 for the remaining values of n shows that the true value falls between $n = 0$ and $n = 0.5$. Values of the order of magnitude of $n = 0.01$ leads to indeterminate solutions for the remaining constants in the equation. This result is to be expected for in these special cases the value of the exponential factor (e^{-bx^n}) tends to approach a constant or a straight line. In the following final computations, the value $n = 1$ has been selected as a sufficiently close approximation to the true value.

TABLE 3.—CONSTANTS FOR TABLE 2

	IX	IX	III	I
(1)	(2)	(3)	(4)	(5)
Σy_o	329.3	329.3	329.3	329.3
Σy_c	337.5	337.0	336.7	336.8
Σv^2	59.26	57.11	56.86	58.41
$\log_e a$	1.23530	1.46607	2.59166	2.59092
m	0.93588	0.81369	....	....
b	0.08033	0.07470	0.07470	....
m_1	....	....	0.81373	0.76915
m_2	....	....	....	15.48968
c_1	....	....	12.0	12.0
c_2	....	....	....	222.0
Origin	1890	1890	1902	1902
Mode	11.65	10.89	−1.11	−0.93
Mode—year	1901.65	1900.89	1900.89	1901.07

The constants of the equation,

$$y = ax^m e^{-bx}$$

are easily determined by writing the observation equations in the form

$$\log_e a + \log_e x\, m - xb = \log_e y.$$

As previously stated, theoretical considerations show that each observation equation must be multiplied by the corresponding value of y_0. Using the data contained in the first two columns of Table 2, we obtain 33 weighted observation equations of the following form:

$$2.4 \log_e a + 0.000\, m - 2.4\, b = 2.101$$
$$3.8 \log_e a + 2.634\, m - 7.6\, b = 5.073$$
$$6.4 \log_e a + 22.378\, m - 211.2\, b = 11.880,$$

the solution of which leads to the results summarized in the second column of table 3. The first derivative vanishes at the points

$$x = \infty \text{ and } x = \frac{m}{b}.$$

The latter point is designated *mode* by Pearson, and *peak* by the petroleum geologist.

The ordinates in the preceding computation were placed at the ends of the year intervals. Precision is generally supposed to be gained by placing the ordinates at the mid-points instead of the end points. The results of making the computation in this way are summarized in table 2, columns 5, 6, and 7; and in table 3, column 3. In this particular case, the value of Σv^2 is reduced from 59.26 to 57.11, but it must not be inferred that a reduction of this character will always be obtained, for the value of Σv^2 is dependent also upon the point selected for origin.

Pearson places the initial origin at any convenient point, and then solves for c_1, the point at which the curve touches the x axis. The corresponding procedure by the method of least squares consists in solving a differential equation similar to (1) for the differential elements da, db, dc, and dm. I have not carried out the computations for this solution. The results obtained by transferring the origin to the point, $x = 12.0$, and adjusting type III on the basis of unrestricted constants are tabulated in columns 9 and 10 of table 2. Comparison of the values summarized in columns 3 and 4 of table 3 shows that the two methods lead to identical results. The numerical values are represented graphically by the curve designated, type IX, in figure 3.

Two attempts have been made to apply Pearson's method of moments: first, to the incomplete data (y_o) tabulated in table 2, column 2; and, second, to assumed complete data obtained by using the observed values (y_o) (column 2, table 2) from 1890 to 1923, inclusive, and the computed values represented by that portion of the type IX curve, figure 3, which extends from 1923 to the end of the curve. The values of κ in the respective cases are $\kappa = -0.0123$ and $\kappa = -0.2240$, thus indicating that type I should be used in preference to type III. Both types fail for incomplete data. Evaluation of the constants from the so called complete data leads to the following results:

Type I. $y = 13.36\left(1 + \frac{x}{8.1240}\right)^{0.4543}\left(1 - \frac{x}{216.37}\right)^{12.1008}$

Mode-year = 1889.5 + 10.016 = 1889.516 Skewness = 0.838
Mean-year = 1889.5 + 24.323 = 1913.823 $\Sigma v^2 = 63.18$
Standard deviation = 17.07 $\Sigma y_c = 328.1$

Type III. $y = 12.42\left(1 + \frac{x}{16.003}\right)^{1.4749} e^{-0.0922x}$

Mode-year = 1889.5 + 13.473 = 1902.973 Skewness = 0.636
Mean-year = 1889.5 + 24.323 = 1913.823 $\Sigma v^2 = 79.22$
Standard deviation = 17.07 $\Sigma y_c = 323.3$

Type I appears to be preferable to type III. As the labor involved in actually deciding the case would be very great, I have evaluated, instead, the constants of type I, using the method of least squares. Three computations were made, assuming for c_1 the arbitrary value, 12.0, and for c_2 the successive values, 132.0, 222.0, and 312.0. The respective values of Σv^2, wherein v was carried to tenths of a unit, were found to be 60.93, 58.41, and 57.97. The curve which represents Σv^2 expressed as a function of c_2 varies so slowly that these values alone do not lead to an accurate determination of the desired constant. However, the value $c_2 = 222.0$ appears to be a close approximation to the true value, and as it does not differ greatly from the value $c_2 = 216.4$ calculated by the method of moments, it has been used in extending the computed curve until it reaches the x axis. The extension is designated type I in figure 3. Over the remaining portion of the curve, the three types of formulae lead to almost identical results, as may be readily verified by comparison of columns 6, 9, and 11 of table 2. The values of Σv^2 tabulated in table 3 differ slightly, however, and suggest that types III or IX should be used in preference to type I, but the agreement in the values is so close, and departures from rigorous methods of evaluation of the constants have been so marked, that definite conclusions can not be drawn except to say that each of the formulas in question gives a very satisfactory representation of the data of observation.

SOME GENERAL RESULTS AND SUGGESTIONS

Table 4 summarizes the results of adjusting the petroleum production data given by Richardson[9] for three states and one western field. The constants of the equation

$$y = ax^m e^{-bx} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

were evaluated by the method of least squares from weighted observation equations of the general form

$$y_o \log_e a + y_o \log_e x\, m - xy_o b = y_o \log_e y_o.$$

The resulting values of the constants are tabulated in columns 1, 2, and 3. Column 4 contains values of r, the probable error of an observation of weight unity. The origins were determined by inspection. A correct determination of these points would reduce slightly

TABLE 4.—SUMMARY OF COMPUTATIONS FOR STATES AND FIELDS

STATE OR FIELD	$\log_e a$	m	b	r	ORIGIN	MODE = $\frac{m}{b}$	Σy_o	Σy_c	$\frac{\Sigma y_o - \Sigma y_c}{\Sigma y_o}$	Σv_m	Σv_1	$\frac{\Sigma v_m}{\Sigma v_1}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
					year							
Ohio	0.82158	1.48136	0.12084	1.32	1884	12.26	513.9	499.4	0.029	169.11	565.1	29.9
Pennsylvania	1.99230	0.92101	0.08595	2.33	1874	10.72	737.4	699.1	0.055	203.74	792.4	25.7
West Virginia	1.46607	0.81369	0.07470	0.93	1890	10.89	337.0	329.3	0.023	110.07	447.7	24.6
Kern River, Cal.	2.42293	0.50828	0.10208	1.34	1900	4.98	251.5	241.3	0.042	63.18	312.4	20.2

the values of r. The mode-year is obtained by addition of the mode to the year selected as origin—thus, the mode-year or peak-year for the state of Ohio is 1884 + 12.26 = 1896.26.

Pearson's method of moments equalizes the observed and computed areas when applied to complete data. The value, $\Sigma y_c = 328.1$, obtained by the method of moments from the assumed complete data of West Virginia is in close agreement with the observed value, $\Sigma y_o = 329.3$; on the other hand, all of the values resulting from the application of least squares to the particular cases discussed in this paper are too large. The magnitude of the discrepancy is shown in column 9 of table 4. The error, for example, in computing the total production from 1890 to 1923 inclusive, for the state of West Virginia, is 2.3 per cent of the observed total production (Σy_o) for the same time interval. The corresponding error resulting from an imperfect application of the method of moments is only 0.4 per cent. Unfortunately, this advantage is offset to a considerable extent by the failure of the method to minimize Σv^2; the difference between the two values, 63.18 and 57.11, is altogether too large. Mathematicians seem not to have settled the question as to the precise relation between the two methods of evaluation of constants. Jackson[11] has shown that the methods are identical

[11] DUNHAM JACKSON. *The method of moments.* Am. Math. Monthly **30**: 307–311. 1923.

for power series, but the relationship existing between the two methods when applied to the more complicated functions has not been established. It is obvious from elementary considerations that the condition of maximum probability and the other condition of equalization of observed and computed areas are not necessarily satisfied at the

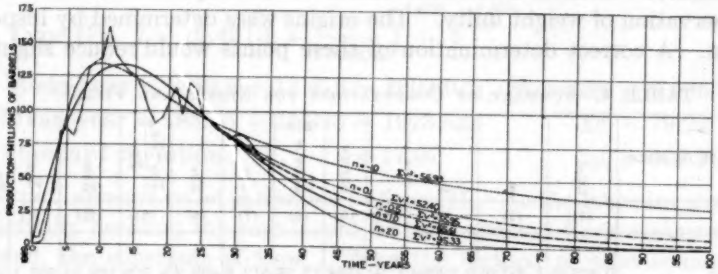

Fig. 2. Curves representing type IX for different values of n

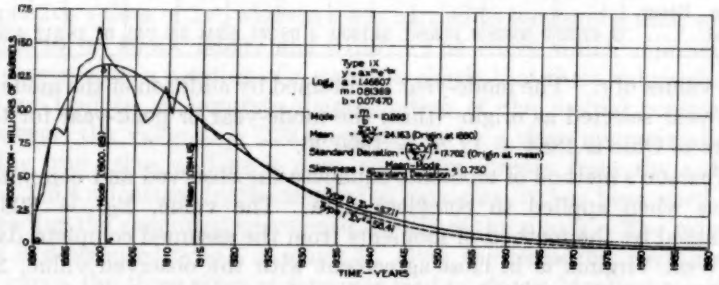

Fig. 3. Curves showing final adjustment of types I and IX

same time. The computations summarized in columns 3 and 8 of table 1 provide a numerical proof of this proposition.

The total area of the computed curve is easily obtained from the formula

$$\text{area} = \int_0^\infty a\, x^m e^{-bx}\, dx = \frac{\Gamma(m+1)}{b^{m+1}} = \frac{m!}{b^{m+1}}$$

The results obtained by substitution in this formula are tabulated in column 11 of table 4. The values, 565.3, 792.1, 448.0, and 312.7 were obtained by summation of the computed ordinates (y_c). The agreement between the two summations is all that could be desired.

As the area of the computed curve from the origin to the mode (area *abca*, figure 3) can not be evaluated by integration; it is necessary to base the calculation on the computed values of y. The areas tabulated in column 10 represent these summations for the integral years and for the fractional part of the year interval extending from the end of the last integral year to the mode. The ratio of computed peak production to computed total production is tabulated in column 12.

The preceding results show that the simple equation (2) can be used to represent a very considerable range of production data with a degree of accuracy that is quite sufficient for most purposes. The constants are easily determined by the method of least squares, and the mode is given by the simple ratio, m/b. The computations represented by the curves in figure 3 show that equation I also gives a satisfactory representation of the data. For other states or areas, some of the remaining equations in the list will undoubtedly prove to be useful.

Pearson's types of frequency distribution promise to be of great value in various fields of science. It may be of interest in this connection to call attention to the possibility that certain sands, for example, may be characterized by the mode, mean, and skewness (see figure 3) of the curve which represents the distribution of grain diameters, and that certain properties of the sands, such as permeability, may be related to these characteristics. The results recently obtained by Wightman, Trivelli, and Sheppard[12] on the relation of size-frequency distribution of grains in photographic emulsions to sensitometric characteristics serve to emphasize the importance of this suggestion.

ACKNOWLEDGMENTS

The investigation on which this paper is based was undertaken at the suggestion of Mr. G. B. Richardson, Geologist, of the U. S. Geological Survey. Two of his assistants, Miss C. M. Shanner and Mrs. L. Boswell assisted in making some of the preliminary computations. The remainder of the computations were made by Mr. C. R. Randall, Junior Geophysicist of the Geological Survey. I am greatly indebted to all of those who have thus contributed so generously and efficiently in the preparation of my paper.

[12] E. P. WIGHTMAN, A. P. H. TRIVELLI, and S. E. SHEPPARD. *The size-frequency distribution of grains of silver halide in photographic emulsions and its relations to sensitometric characteristics.* Abridged Scientific Publications from the Research Laboratory of the Eastman Kodak Company 7, no. 172: 24–33. 1923.

PHYSICS.—*Note on the theory of optical dispersion.* G. BREIT, Department of Terrestrial Magnetism. (Communicated by L. A. BAUER.)

Soon after the development of the $\int p dq = nh$ principle attempts have been made by Sommerfeld and Debye[1] to account for optical dispersion by studying the perturbations caused in the electronic motion by the incident wave. These attempts, however, were unsuccessful because the frequency at which the calculation showed the reaction to be strongest proved to be the frequency of revolution of the electron and not the absorption frequency given by the relation $\Delta W = h\nu$. This difficulty was avoided by Ladenburg and by Ladenburg and Reiche[2] by disregarding the nature of the perturbations and by postulating directly that an atom which is in a position to absorb radiation of frequency ν_a contributes to the refractive index of the medium in which it is situated a term which would be contributed by a linear oscillator of the same frequency ν_a. A modification of this theory has been given by Kramers.[3]

In these theories the dispersion is discussed only for frequencies which do not lie in the absorption band. For this reason the equivalent classical resonator is not supposed to have any dissipation of energy. It is the purpose of the present note to discuss how far it is necessary to introduce the dissipation of energy into the theory.

On the classical theory a frictional term in the equations of motion could be interpreted as due to radiation. On the quantum theory there is no continuous radiation. Thus a frictional term introduced directly into the equations of motion of the classical resonator would appear, to a certain extent, illogical, and one is led to look more closely into the possibilities of a classical resonator without a frictional term We adopt the notation of Van Vleck,[4] and consider an energy level ε from which it is possible for the atom to go into higher states r by absorbing energy and to descend into lower states t by emitting energy Taking the orientations of the atoms to be random, the polarization for N_s electrons in the state s is hypothesized by Kramers to be

$$P_s = \frac{e^2}{32\pi^4}\left[\Sigma_r \frac{A_{r\to s}}{\nu_{rs}^2(\nu_{rs}^2-\nu^2)} - \Sigma_t \frac{A_{s\to t}}{\nu_{st}^2(\nu_{st}^2-\nu^2)}\right] N_s \mathcal{E}$$

[1] DEBYE, Sitzungsber. Münchener Akademie, p. 1. 1915.

[2] LADENBURG and REICHE, Naturwissenschaften **27**: 584. July 6, 1923.

[3] H. A. KRAMERS, Nature, May 10, 1924, p. 673; August 30, 1924, p. 310. See also VAN VLECK, Phys. Rev. **24**: 344. 1924.

[4] J. H. VAN VLECK, Phys. Rev. **24**: 330. 1924.

[5] VAN VLECK, loc. cit. Equation 17.

where $A_{r\to s}dt$ is the *a priori* probability that in the time dt an atom which is in the state r should fall into the state s. Whether this is the actual form or only similar to reality the essence of what follows remains unaffected.

Let us consider one particular dispersion term.

$$P_{rs} = \frac{c^3}{32\pi^4} \frac{A_{rs}}{\nu_{rs}^2 (\nu_{rs}^2 - \nu^2)} N_s \mathscr{E} \tag{1}$$

This means that under the action of the incident wave having an intensity $\mathscr{E}$ the N_s atoms which are in the state s suffer a total polarization P_{rs} owing to their ability to be transferred to the state r. Now it is clear that any single atom can not give a purely monochromatic expression (1) because it does not remain in the state s indefinitely but on account of the $A_{s\to t}$ is bound to leave s spontaneously. Therefore, even if one should suppose that while an atom is in the state s it gives a polarization of the type (1), one must expect an expression different from (1) for the observed average polarization. Expression (1) is essentially the reaction on the wave due to a classical oscillator and for this reason we shall simplify the discussion by considering a resonator having the equation of motion[6]

$$\ddot{x} + \Omega^2 x = \frac{e}{m} E_x \tag{2}$$

Here E_x is the x component of the incident wave, e, m, Ω are respectively the charge, mass, and resonant frequency of the resonator. The incident wave we suppose to be of the type

$$E_x = E_{x0} \cos(\omega_0 t - \epsilon) \tag{3}$$

Let us suppose that at the time $t = t_1$, the atom falls into the state s and that at the time $t = t_2$ it falls out of it. It is impossible to say (with the present state of our knowledge) what the initial conditions of the oscillator (2) are at $t = t_1$. Lacking better information, however, we may take the oscillator to be in a state of rest and in a position of equilibrium at the instant of its birth. The solution of (2) with these initial conditions is

[6] While writing this note the author was unaware of the derivation given by H. A. LORENTZ (Proc. Amst. Acad. 8: 591. 1906.) in which formula (10) is derived. Inasmuch, however, as the present note considers the question from the point of view of Fourier integrals, and shows the manner in which only the part of the polarization synchronous with the incident wave survives, it has not been withdrawn from publication.

$$x = \frac{e\,E_{xo}}{m\,(\Omega^2 - \omega_0^2)}\left[-\cos(\epsilon - \omega_0 t_1)\cos\Omega(t - t_1) - \frac{\omega_0}{\Omega}\sin(\epsilon - \omega_0 t_1)\sin\Omega(t - t_1) + \cos(\omega_0 t - \epsilon)\right] \quad (4)$$

In the neighbourhood of $\omega_0 = \Omega$ this is very nearly the same as

$$x = \frac{e\,E_{xo}}{m\,(\Omega^2 - \omega_0^2)}\left[\cos(\omega_0 t - \epsilon) - \cos\{\Omega(t - t_1) + \omega_0 t_1 - \epsilon\}\right] \quad (4')$$

Letting

$$t_2 - t_1 = T \quad (4.5)$$

and neglecting $\omega - \Omega$ in comparison with Ω we have in the neighbourhood of the absorption frequency the following Fourier Integral for x

$$x(t; t_1, T) \doteq \frac{e\,E_{xo}}{\pi\,m\,(\Omega^2 - \omega_0^2)}\int_0^\infty \left\{\left[\frac{\sin(\omega - \omega_0)T}{2(\omega - \omega_0)} - \frac{\sin(\omega - \Omega)T}{2(\omega - \Omega)}\right]\cos(\omega t - \epsilon - (\omega - \omega_0)t_1) + \left[\frac{\sin^2\left(\frac{\omega - \omega_0}{2}T\right)}{\omega - \omega_0} - \frac{\sin^2\left(\frac{\omega - \Omega}{2}T\right)}{\omega - \Omega}\right]\sin(\omega t - \epsilon - (\omega - \omega_0)t_1)\right\} d\omega \quad (5)$$

Let us suppose that the intervals (t_1, t_2) are distributed throughout a long interval $\mathcal{T}$ in such a way that on the average their density is 1. The value which concerns us is

$$\int_{t_1=0}^{\mathcal{T}} x(t; t_1, T)\,dt_1 \doteq$$

$$\frac{e\,E_{xo}}{\pi m(\Omega^2 - \omega_0^2)}\int_0^\infty \left\{\frac{\sin(\omega - \omega_0)\mathcal{T}}{\omega - \omega_0}\left[\left(\frac{\sin(\omega - \omega_0)T}{2(\omega - \omega_0)} - \frac{\sin(\omega - \Omega)T}{2(\omega - \Omega)}\right)\cos(\omega t - \epsilon) + \left(\frac{\sin^2\left(\frac{\omega - \omega_0}{2}T\right)}{\omega - \omega_0} - \frac{\sin^2\left(\frac{\omega - \Omega}{2}T\right)}{\omega - \Omega}\right)\sin(\omega t - \epsilon)\right] + \frac{2\sin^2\left(\frac{\omega - \omega_0}{2}\mathcal{T}\right)}{\omega - \omega_0}\left[\left(\frac{\sin(\omega - \omega_0)T}{2(\omega - \omega_0)} - \frac{\sin(\omega - \Omega)T}{2(\omega - \Omega)}\right)\sin(\omega t - \epsilon) -\right.$$

$$\left(\frac{\sin^2\left(\frac{\omega-\omega_0}{2}T\right)}{\omega-\omega_0}-\frac{\sin^2\left(\frac{\omega-\Omega}{2}T\right)}{\omega-\Omega}\right)\cos(\omega t-\epsilon)\Bigg]\Bigg\}\,d\omega \qquad (6)$$

Again, to within the same approximation, the wave (2) when acting for a time $\mathscr{T}$, and analyzed into a Fourier Integral is

$$E_x \doteq \frac{E_{x0}}{\pi}\int_0^\infty \frac{\sin(\omega-\omega_0)\mathscr{T}}{2(\omega-\omega_0)}\cos(\omega t-\epsilon)d\omega \qquad (7)$$

Since $\mathscr{T}$ must be thought of as $\gg T$ the band of frequencies represented by $\frac{\sin(\omega-\omega_0)\mathscr{T}}{\omega-\omega_0}$ is to be thought of as infinitely more sharp than that represented by $\frac{\sin(\omega-\omega_0)T}{\omega-\omega_0}$, and consequently we may rewrite (6) as

$$\int_{t_1=0}^{\mathscr{T}} x(t;t_1T)\,dt_1 \doteq$$

$$\frac{e\,E_{x0}}{\pi m(\Omega^2-\omega_0^2)}\int_0^\infty\Bigg\{\frac{\sin(\omega-\omega_0)\mathscr{T}}{\omega-\omega_0}\Bigg[\left(\frac{T}{2}-\frac{\sin(\omega_0-\Omega)T}{2(\omega_0-\Omega)}\right)\cos(\omega t-\epsilon)-$$

$$\frac{\sin^2\left(\frac{\omega_0-\Omega}{2}T\right)}{\omega_0-\Omega}\sin(\omega t-\epsilon)\Bigg]$$

$$+\frac{2\sin^2\left(\frac{\omega-\omega_0}{2}\mathscr{T}\right)}{\omega-\omega_0}\Bigg[\left(\frac{T}{2}-\frac{\sin(\omega_0-\Omega)T}{2(\omega_0-\Omega)}\right)\sin(\omega t-\epsilon)+$$

$$\frac{\sin^2\left(\frac{\omega_0-\Omega}{2}T\right)}{\omega_0-\Omega}\cos(\omega t-\epsilon)\Bigg]\Bigg\}\,d\omega \qquad (8)$$

Since $\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x}\,dx = 0$ we may disregard the part of (8) containing $\frac{\sin^2\left(\frac{\omega-\omega_0}{2}\mathscr{T}\right)}{\omega-\omega_0}$. Comparing what is left of (8) with (7) it is clear that the atom acts as though during the time $\mathscr{T}$ it had a polarization

$$P = \frac{e\,E_\infty}{m(\Omega^2 - \omega_0^2)} \left\{ \left(T - \frac{\sin(\omega_0 - \Omega)T}{\omega_0 - \Omega} \right) \cos(\omega_0 t - \epsilon) - \frac{2\sin^2\left(\frac{\omega_0 - \Omega}{2} T\right)}{\omega_0 - \Omega} \sin(\omega_0 t - \epsilon) \right\} \tag{9}$$

If now we should suppose that the times T have a distribution function of the type $e^{-\left(\frac{T}{T_0}\right)} \mathrm{d}\left(\frac{T_0}{T}\right)$ then we obtain an average value of P from (9) *viz.*

$$P = \frac{e\,E_\infty\,T_0^2\,(\Omega - \omega_0)\,[T_0 \cos(\omega_0 t - \epsilon) + (\Omega - \omega_0)^{-1} \sin(\omega_0 t - \epsilon)]}{m\,(\Omega + \omega_0)\,[1 + T_0^2\,(\omega_0 - \Omega)^2]} \tag{10}$$

This result for the polarization is thus obtained on the hypothesis of intermittent actions, the number of intervals of lengths between T and $T + \mathrm{d}T$ beginning per second being $e^{-\frac{T}{T_0}} \mathrm{d}\,\frac{T}{T_0}$. The total number of intervals started per second is therefore 1 and the average duration of an interval is T_0. Hence, if we are to compare (10) with the result of the classical theory for a single continuously acting resonator having a frictional term then we must divide (10) by T_0. To within the approximations made (on performing an elementary calculation for the classical resonator) the result is seen to be identical on the two theories.

We have thus shown that the phase change introduced by the classical frictional force may be also thought of quantitatively as due to the interrupted action of the resonator. If the Correspondence Principle may be applied in this case and if it is to be interpreted as assigning a certain similarity to the actual dispersion and the theoretical classical dispersion then we are practically forced to abandon any other influence than that of interrupted actions in the higher quantum states. If, further, the width of emission and absorption lines is related to the dispersion by the laws of the classical theory then we must think of this width as being entirely due to the finite time of each independent emission.

Absorptions beginning in the lowest energy level occupy a singular position in this theory because atoms may remain in these indefinitely. Whether they suffer phase changes in their motion while in the lowest state is unknown but for the sake of uniformity such changes might be hypothesized.

It is of interest to observe (as will be brought out in more detail in another note) that the above theory of interruptions accounts for the polarization of resonance radiation in weak magnetic fields in a more satisfactory manner than the classical resonator.[7]

ENTOMOLOGY.—*New mosquitoes from Brazil (Diptera, Culicidae).* HARRISON G. DYAR and RAYMOND C. SHANNON, U. S. National Museum. (*Communicated* by S. A. ROHWER.)

Dr. J. Bequaert has placed before us for determination a collection of mosquitoes recently made in Brazil, chiefly in the State of Amazonas. Among a number of interesting species collected, the following are hitherto undescribed.

Sabethoides glaucodaemon, new species.

Of usual size in the genus, largely purplish black; proboscis extending well beyond the antennae, slightly longer than the abdomen; palpi small, slightly longer than the two basal flagellar joints; eyes contiguous on lower side of head for a greater distance than they are above; prothoracic lobes contiguous above, their scales overlapping; rather numerous setae on anterior margin; mesonotum with setae only above roots of wings and a few on anterior margin; spiracular sclerite with three setae; propleura with two setae; sternopleura without setae; a small but dense tuft of long setae on upper posterior corner of mesepimeron; pleurae with dense white scales below; trochanters and base of femora yellow, the under sides of the femora white scaled basally along the entire length of the posterior pair; abdomen compressed laterally, dark scaled on upper half, yellowish white scaled below, the colors divided in a straight line, though the white is illy contrasted in certain lights. Wings normal, basal cross-vein opposite the anterior cross-vein; roots of halteres yellow, stem and knob blackish; mid tarsi white scaled below on last four joints except narrowly at base of second.

Type, female, no. 27744, U. S. Nat. Mus.; near San Alberto, Rio Branco, Amazonas, Brazil, August 28, 1924 (*J. Bequaert*).

Nearest related to *imperfectus* B.-W. & B.,differing chiefly in the slightly longer proboscis, and from both this and *chloropterus* Humb. in the coloration of the abdomen.

Culex (Choeroporpa) bequaerti, new species

Rather small dark brown species; occiput with erect forked scales, all the recumbent scales broad, mostly white in front, a patch of black ones on each side of the middle; antennae fairly long, exceeding length of proboscis, which is about equal to the length of abdomen; integument of mesonotum very dark brown, scutellum somewhat paler; scales narrow, dark brown; dorsal setae sparse but well developed; a row of pronotal setae; pleurae yellowish,

[7] In such a way we may think of the field caused by virtual oscillators as virtual in the sense that it carries energy away only statistically but does not necessarily produce damping of the motions induced in the virtual oscillator by the exciting field.

prealar setae about seven, a row of fairly strong setae along posterior margin of sternopleura, a little weaker below; a single lower mesepimeral seta; legs entirely dark, except for paler ventral surfaces of femora; abdomen dark brown, with whitish scales on ventral surface at bases of segments; wing scales broad subcostally, those on base of fork of second vein narrowly ovate to ligulate; halteres pale, the knobs dark.

Male: palpi exceeding the proboscis by nearly the length of the last two joints, the penultimate joint with a small whitish ring at base, otherwise dark. Scales of mesonotum deep bronzy brown; three pronotal setae, broken in the female type.

Hypopygium.—Side piece a little longer than hemispherical; inner division of lobe strong, running far into the side piece, with an infuscated patch basally, columnar, long, exceeding the outer lobe, with two strong long hooked and distorted filaments at tip, one inserted basally of the other; outer division small, with four stout rod-like filaments on the oblique outer aspect and a small rounded leaf basally of them. Clasper slenderly snout-shaped, the spine appendiculate. Tenth sternites comb-shaped, with six teeth, enlarged at base, with only a rudiment of basal projection; first plate of mesosome normal, the articulated plate rather narrow, emarginate on one side; second plate curved, tip furcate, the arms short, inner pointed, outer smooth, a long strong horn a little beyond the middle of the stem; basal hooks slender, strongly recurved, not projecting at base; ninth tergites conically pointed, small, setose, connected by a chitinous band.

Type, male, paratype, female, no. 27745, U. S. Nat. Mus.; male, Sororoca, Rio Branco, Amazonas, Brazil, September 1, 1924; female, Carmo, with other data the same (*J. Bequaert*).

The preceding description was written before we had an opportunity of examining Miss A. M. Evans' recent paper[1] describing new *Choeroporpa* from Brazil. Of the species there described *C.* (*C.*) *thomasi* Evans comes nearest to the present form. The mesosomal plate in the two is much the same. The comb of the tenth sternite of *thomasi* appears abnormal in Miss Evans' figure; in *bequaerti* it consists of seven long equal teeth. The inner division of the lobe of the side piece has a longer stem in *bequaerti* than in *thomasi*, the distance between the insertion of the two filaments being less than the remaining basal part of the stem, whereas in the figure of *thomasi* the reverse is the case. The outer division of the lobe of the side piece is differently formed, having no inner limb in *bequaerti* and the leaf is inserted on the stem basally of the other filaments, whereas in *thomasi* it arises between the limb and the outer setal group.

Culex (Choeroporpa) albinensis Bonne-Wepster & Bonne

Culex (*Choeroporpa*) *albinensis* Bonne-Wepster & Bonne, Ins. Ins. Mens. **7**: 173. 1920.
Culex (*Choeroporpa*) *albinensis* Dyar, Ins. Ins. Mens. **8**: 62. 1920.
Culex (*Choeroporpa*) *gordoni* Evans, Ann. Trop. Med. & Par. **18**:369. 1924.

No difference is apparent between Miss Evans' detailed figures of the

[1] Ann. Trop. Med. and Par. **18**: 363–375. 1924.

hypopygium of *gordoni* and the characters of *albinensis* in a slide before us. The distribution of the species includes Paramaribo, Surinam, and Manáos, Brazil.

Mansonia indubitans, new species

Basal antennal joints as dark as rest of antennae; antenna somewhat shorter than length of proboscis; palpi as long as four basal flagellar joints; proboscis on basal two-fifths with pale and dark scales intermixed, a rather broad white ring a little beyond middle, beyond blackish, paler at apex; mesonotum dark brown, setae normal, sparse small narrow golden scales intermixed, with long dark scales on the sides posteriorly; a number of pronotal setae; postspiracular setae present; sterno-pleural on posterior margin with one long stout seta midway and smaller setae on either side, mesepimeron with three setae near anterior margin; femora and tibiae with dark and light scales intermixed; hind tibiae darker; first tarsal joint without basal or apical ring, but with scattered white scales on inner surface; second and third tarsal joints white basally, also the fourth joint of mid and hind tarsi, remainder dark; wings dark scaled with numerous white ones intermixed, all broad; abdomen dark, with triangular patch of white scales on first segment, apex directed forward, the venter with numerous broad white scales intermixed. Knobs of halteres dark brown.

Type, female, paratypes, three females, no. 27746, U. S. Nat. Mus.; Belem, Para, September 19, 1924; Carmo, Rio Branco, Amazonas, August 31, 1924; above Santarem, July 22, 1924, Itacoatiara, Amazon River, September 15, 1924 (*J. Bequaert*).

Similar to *titillans* Walker, the palpi shorter, and with slight differences in coloration as indicated above.

Anopheles celidopus, new species

Medium size, grayish in general appearance; occiput with erect truncate white scales above, dark brown below, white setae and scales between eyes; antenna shorter than palpus, scales only on the basal flagellar joint; palpus but little shorter than proboscis, with outstanding dark brown scales, a few white ones at apices of second and third joints; prothoracic lobes with tuft of scales above; mesonotum with pale curved hairs sparsely distributed, a little denser anteriorly and darker on the sides; pleura with two indefinite pale pollinose lines; legs dark with narrow white rings at apices and bases of all but the last tarsal joints; abdomen dark, with sparse dark hairs, a few white scales on the dorsum of the last segment, and many dark and white scales on the venter of this segment; cerci densely scaled, mostly dark; wings with eight more or less definite white spots on anterior margin, and numerous other small white spots irregularly distributed over the wing; wing scales lanceolate; knobs of halteres dark.

Type, female, paratypes, four females, no. 27747, U. S. Nat. Mus.; Carmo, Rio Branco, Amazonas, Brazil, September 1, 1924 (*J. Bequaert*).

This species does not fit well into any of the existing groups of *Anopheles*. It comes nearest to *Arribalzagia*, but lacks the lateral scale tufts of the abdomen.

SCIENTIFIC NOTES AND NEWS

DR. S. G. MORLEY, Associate in Middle American Archaeology, Carnegie Institution of Washington, gave public lectures at the Institution on October 28 and December 3, on *Initial excavations at Chichen Itza, Yucatan.* Under agreement with the government of Mexico, the Institution has the privilege of excavation and archaeological study at this ancient Maya city for a period of ten years, beginning January, 1924.

DR. A. E. DOUGLASS, Director of the Steward Observatory, University of Arizona, gave an illustrated lecture at the Carnegie Institution of Washington on December 5, on *Cycles in development of tree rings as an evidence of climatic variation.*

Two lectures on atmospheric dynamics were given at the Carnegie Institution on January 6 and 8, by PROF. V. BJERKNES, professor of hydrody. namics at the Geophysical Institute at Bergen, Norway, and research associate of the Institution. The first lecture was on *The forces which lift airplanes,* and the second, on *Problems in dynamical meteorology.*

DR. J. A. ANDERSON, of the Mt. Wilson Observatory of the Carnegie Institution, gave an illustrated lecture at the auditorium of the Institution on January 5, under the title: *An experimental method of studying high temperatures.*